9. Continuity and Uniform Continuity

We will now shift our focus over to functions. We will discuss continuity and uniform continuity, along with some supplemental theorems.

Definition 1 (Continuity)

Let E be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$. f is said to be continuous at a point $a \in E$ if and only if given $\varepsilon > 0$ there is a $\delta > 0$ (which in general depends on ε, f, and a) such that $|x - a| < \delta$ and $x \in E$ implies that $|f(x) - f(a)| < \varepsilon$
f is said to be continuous on E if and only if f is continuous at every $x \in E$.

We shall now explore two theorems:

Theorem 1 (Extreme Value Theorem)

If I is a closed, bounded interval and $f : I \to \mathbb{R}$ is continuous on I then $f$is bounded on I. Moreover, if $M = \sup_{x \in I} f(x)$ and $m = \inf_{x \in I} f(x)$ then there exist points $x_m, x_M \in I$ such that $f(x_M) = M$ and $f(x_m) = m$.

Theorem 2 (Intermediate Value Theorem)

Suppose that $a < b$ and $f : [a, b] \to \mathbb{R}$ is continuous. If y_0 lies between $f(a)$ and $f(b)$ then there is an $x_0 \in (a, b)$ such that $f(x_0) = y_0$.

The following function is a common function to use as a counterexample as it is discontinuous everywhere. These functions are called nowhere continuous.

Definition 2 (Dirichlet Function)

The Dirichlet function is defined on $\mathbb{R}$ by,

$$f(x) := \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

We shall now introduce uniform continuity:

Definition 3 (Uniform Continuity)

Let E be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$. Then f is said to be uniformly continuous on E if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|x - a| < \delta$ and $x, a \in E$ implies $|f(x) - f(a)| < \varepsilon$.

Notice the key difference between uniform continuity and the standard continuity—the δ in the definition of uniform continuity does not depend on a, whereas the δ in the definition of continuity is allowed to vary based on a.
We shall finally introduce an enrichment topic – Lipschitz continuity:

Definition 4 (Lipschitz Continuity)

Let E be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$. Then f is said to be Lipschitz continuous on E if and only if there exists a constant $L > 0$ such that $|f(x) - f(a)| \leq L|x - a|$ for all $x, a \in E$. Any such constant L is called a Lipschitz constant for f.