8. Doubles Sequences and Their Limits

We shall now do some enrichment material on double sequences and their limits.
We shall start our discussion with a few definitions of double sequences and the different types of limits we can take.

Definition 1 (Double Sequence)

A double sequence is a function $a : N \times \mathbb{N} \to \mathbb{R}$ denoted by $(a_{n,m})$. For each $(n, m) \in \mathbb{N} \times \mathbb{N}$, $a_{n,m}$ represents a real number.

Definition 2 (Double Limit)

Let $(a_{n,m})$ be a double sequence. For some $L \in \mathbb{R}$, we say L is the double limit of $(a_{n,m})$, or equivalently that $a_{n,m} \to L$ as $(n, m) \to \infty$ if and only if for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|a_{n,m} - L| < \varepsilon$.

Definition 3 (Proper Divergence of Double Sequences)

We say that a double sequence $(a_{n,m})$ is properly divergent if $\lim_{(n,m)\to\infty} a_{n,m} = \pm\infty$.

Definition 4 (Iterated Limits)

For a double sequence $a_{n,m}$, we define two iterated limits (when they exist).

(a) We say $\lim_{n\to\infty}(\lim_{m\to\infty} a_{n,m}) = L_1$ means that for each fixed n, $\lim_{m\to\infty} a_{n,m}$ exists, call it b_n, and that $\lim_{n\to\infty} b_n = L_1$.

(b) We say $\lim_{m\to\infty}(\lim_{n\to\infty} a_{n,m}) = L_2$ means that for each fixed m, $\lim_{n\to\infty} a_{n,m}$ exists, call it c_m, and that $\lim_{m\to\infty} c_m = L_2$.

We shall quickly state that the double limit is unique:

Theorem 1 (Uniqueness of Double Limits)

A double sequence $(a_{n,m})$ can have at most one double limit.

Proof. Suppose that L, L' are both limits of $(a_{n,m})$. Then given $\varepsilon > 0$ there exist natural numbers N_1, N_2 such that,

$$n, m \geq N_1 \implies |a_{n,m} - L| < \frac{\varepsilon}{2}$$

and that,

$$n, m \geq N_2 \implies |a_{n,m} - L'| < \frac{\varepsilon}{2}$$

Let $N := \max N_1, N_2$. Then for all $n, m \geq N$, the above two implications yield that $0 \leq |L - L'| = |L - a_{n,m} + a_{n,m} - L'|$. By triangle inequality, we get that this is less than or equal to $|a_{n,m} - L| + |a_{n,m} - L'| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. □

We shall define an equivalence theorem for double limits and iterated limits:

Theorem 2 (Equivalence Theorem for Double Sequences)

Let $(a_{n,m})$ be a double sequence. If the double limit $L := \lim_{(n,m)\to\infty} a_{n,m}$ exists, and for each m the limit $\lim_{n\to\infty} a_{(n,m)}$ exists, and for each n the limit $\lim_{m\to\infty} a_{(n,m)}$ exists, then both iterated limits exist and equal L.

We shall give a brief definition of boundedness and monotonicity:

Definition 5 (Bounded Double Sequences)

A double sequence $(a_{n,m})$ is bounded if there exists some $M > 0$ such that $|a_{n,m}| \leq M$ for all $m, n \in \mathbb{N}$.

Definition 6 (Monotonicity of Double Sequences)

Let $(a_{n,m})$ be a double sequence of real numbers. If $a_{n,m} \leq a_{j,k}$ for all $(n, m) \leq (j, k)$, we say that $(a_{n,m})$ is increasing. If $a_{n,m} \geq a_{j,k}$ for all $(n, m) \leq (j, k)$, we say that $(a_{n,m})$ is decreasing. If $(a_{n,m})$ is either increasing or decreasing, we say it is monotone.

We shall now extend the Monotone Convergence Theorem to double sequences:

Theorem 3 (Monotone Convergence Theorem for Double Sequences)

If a double sequence $(a_{n,m})$ is monotone in both variables and bounded then both iterated limits exist and the double limit exists. Further, all three limits are equal.