7. Limit Theorems, Monotone and Bolzano-Weierstrass

We shall continue our discussion of convergence and sequences.
Here are some useful theorems regarding sequences:

Theorem 1 (Squeeze Theorem for Sequences)

Let $(x_n), (y_n), (w_n) \in \mathbb{R}^{\mathbb{N}}$.

(a) If $x_n \to a$ and $y_n \to a$ as $n \to \infty$ and if there exists an $N_0 \in \mathbb{N}$ such that $x_n < w_n < y_n$ for $n \geq N_0$ then $w_n \to a$ as $n \to \infty$.

(b) If $x_n \to 0$ as $n \to \infty$ and (y_n) is bounded then $x_n y_n \to 0$ as $n \to \infty$.

Theorem 2 (Comparison Theorem for Sequences)

Let $(x_n), (y_n) \in \mathbb{R}^{\mathbb{N}}$ be convergent. Suppose $x_n \to a$ and $y_n \to b$ as $n \to \infty$. If there exists some $N_0 \in \mathbb{N}$ such that $x_n \leq y_n$ for $n \geq N_0$, then $a < b$. In particular, if $x_n \in [i, j]$ then $a \in [i, j]$.

Here are some useful and intuitive definitions.

Definition 1 (Increasing/Decreasing)

(a) We say (x_n) is increasing if $x_1 \leq x_2 \leq \ldots$.

(b) We say (x_n) is decreasing if $x_1 \geq x_2 \geq \ldots$.

(c) We say (x_n) is strictly increasing if $x_1 < x_2 < \ldots$.

(d) We say (x_n) is strictly decreasing if $x_1 > x_2 > \ldots$.

Notationally, if (x_n) is an increasing sequence which converges to a, we typically write $x_n \uparrow a$ as $n \to \infty$. Similarly, if (x_n) is a decreasing sequence which converges to a, we typically write $x_n \downarrow a$ as $n \to \infty$.

Definition 2 (Monotone)

We say that (x_n) is monotone if it is increasing or decreasing, and strictly monotone if it is strictly increasing or strictly decreasing.

Now that we are equipped with these new definitions, we can write several theorems. First, we will show a result for sequences which are bounded and monotone.

Theorem 3 (Monotone Convergence Theorem)

If (x_n) is increasing and bounded above or decreasing and bounded below, then $x_n \to a$ for finite $a \in \mathbb{R}$.

Proof. We shall prove for the case when $(x_n) \in \mathbb{R}^{\mathbb{N}}$ is increasing and bounded above. The other case, where (x_n) is decreasing and bounded below, follows similarly. Assume (x_n) is increasing and bounded above. Then for all $n \in \mathbb{N}, x_n \leq x_{n+1}$ and there exists some $M \in \mathbb{R}$ such that $x_n \leq M$ for all n. Consider $E = \{x_n : n \in \mathbb{N}\}$. We observe that E is nonempty, is bounded above by M, and so by Completeness Axiom, $\sup E$ exists. Let's call this supremum $s := \sup E$. We want to show that for any $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that $|x_n - s| < \varepsilon$ for all $n \geq N$. By the Approximation Property for Suprema, we derive that for all $m \geq n$, $s - \varepsilon < x_m \leq s$. Hence $|x_m - s| < \varepsilon$ for all $m \geq n$. Thus, $x_n \to a$ as $n \to \infty$ for $a \in \mathbb{R}$. □

We shall introduce nested intervals.

Definition 3 (Nested Intervals)

We say a sequence of intervals is nested if each interval is a subset of the previous interval. Specifically, (I_n) is nested if $I_1 \subseteq I_2 \subseteq \ldots$

A useful theorem on nested intervals follows.

Theorem 4 (Nested Interval Property)

If (I_n) is a sequence of nested intervals then,

$$E := \bigcap_{n=1}^{\infty} I_n$$

is nonempty. Further, if $|I_n| \to 0$ as $n \to \infty$ then $E = \{a\}$ for some $a \in \mathbb{R}$.

We shall finally consider the Bolzano-Weierstrass Theorem, which is the ultimate theorem of this section. You may see this pattern throughout mathematics – results which can be stated simply are often difficult to prove. They're also often extremely interesting.

Theorem 5 (Bolzano-Weierstrass Theorem)

Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Proof. Let (x_n) be a bounded sequence in $\mathbb{R}$. Since it is bounded, there exist $M, m \in \mathbb{R}$ such that $m \leq x_n \leq M$ for all n. Let $I_1 = [m, M]$. Divide I_1 into two equal closed intervals, where the left half is $[m, \frac{m+M}{2}]$ and the right half is $[\frac{m+M}{2}, M]$. At least one of these halves must contain infinitely-many terms of (x_n) (by the Pigeonhole Principle). Denote this half I_2. Choose n_1 to be the smallest index such that $x_{n_1} \in I_1$. This will be the first term of our subsequence. Repeat the bisection process on I_2. Again, one half must contain infinitely-many terms. Call this half I_3. Choose $n_2 > n_1$ such that $x_{n_2} \in I_2$. We shall continue this process inductively. At step k we have interval I_k containing infinitely-many terms. Bisect I_k and choose the half with infinitely many terms as I_{k+1}. Select $n_k > n_{k+1}$ such that $n_{n_k} \in I_k$. Observe that $|I_k| = \frac{M-m}{2^{k-1}}$. Note that the intervals are nested, such that $I_{k+1} \subseteq I_k$ for all $k \in \mathbb{N}$. Note that $|I_k| \to 0$ as $k \to \infty$. Thus, by the Nested Interval Property, $\bigcap_{k=1}^{\infty} I_k = \{a\}$ for some $a \in \mathbb{R}$. It follows that $x_{n_k} \to a$ as $k \to \infty$. $\square$