6. Cauchy, Cluster Points and Limits Infimum/Supremum

This week, I will give some depth to our discussion of sequences and convergences. We will discuss Cauchy sequences, cluster points, and limits supremum/infimum.

Definition 1 (Cauchy Sequence)

A sequence is called Cauchy if and only if for all $\varepsilon > 0$ there is some natural number N such that for all $n, m \geq N$, $|x_n - x_m| < \varepsilon$.

Conceptually, this means a sequence is Cauchy if points get closer to each other. Contrast this to a convergent sequence, where points get closer to some limiting value a.
We shall now explore the relationship between Cauchy and convergent sequences in $\mathbb{R}$:

Theorem 1 (Cauchy-Convergence Equivalence in $\mathbb{R}$)

A sequence (x_n) is convergent if and only if it is Cauchy.

Proof. First, we shall prove sufficiency. (Note that this argument works for any sequences, not strictly within $\mathbb{R}$.) Suppose that $x_n \to a$ as $n \to infty$. Then by definition, given $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $|x_n - a| < \frac{\varepsilon}{2}$ for all $n \geq N$. Hence if $n, m \geq N$, it follows from the triangle inequality that $|x_n - x_m| \leq |x_n - a| + |x_m - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.
Now we shall prove necessity. Suppose that (x_n) is a real Cauchy sequence. Given $\varepsilon = 1$, choose $N \in \mathbb{N}$ such that $|x_N - x_m| < 1$ for all $m \geq N$. By the triangle inequality, $|x_m| < 1 + |x_N|$ for $m \geq N$. Therefore (x_n) is bounded by $M := \max\{|x_1|, ..., |x_{N-1}|, 1 + |x_N|\}$. By Bolzano-Weierstrass, (x_n) has a convergent subsequence, say $x_{n_k} \to a$ as $k \to \infty$. Let $\varepsilon > 0$. Since x_n is Cauchy, choose $N_1 \in \mathbb{N}$ such that $n, m \geq N_1$ implies $|x_n - x_m| \leq \frac{\varepsilon}{2}$. Since $x_{n_k} \to a$ as $k \to \infty$, choose $N_2 \in \mathbb{N}$ such that $k \geq N_2$ implies $|x_{n_k} - a| < \frac{\varepsilon}{2}$. Fix $k \geq N_1$ such that $n_k \geq N_2$. Then $|x_n - a| \leq |x_n - x_{n_k}| + |x_{n_k} - a| < \varepsilon$ for all $n \geq N_1$. Thus $x_n \to a$ as $n \to \infty$. □

Notice that in the above proof, when proving necessity, the Bolzano-Weierstrass Theorem required $(x_n) \in \mathbb{R}^{\mathbb{N}}$.
We shall now discuss cluster points:

Definition 2 (Cluster Point of a Set)

Let E be a subset of $\mathbb{R}$. A point $a \in \mathbb{R}$ is called a cluster point of E if $E \cap (a - r, a + r)$ contains infinitely many points for every $r > 0$.

We can similarly define cluster points of a sequence:

Definition 3 (Cluster Point of a Sequence)

A sequence (x_n) has a cluster point $a \in \mathbb{R}$ such that for every open interval I around a, there are infinitely-many natural numbers $n \in \mathbb{N}$ such that $x_n \in I$.

Example 1

Consider $(x_n) = (-1)^n \frac{n}{n+1}$. It has cluster points at -1 and 1.

Now we shall discuss limits infimum and supremum.

Definition 4 (Limit Infimum/Supremum)

The limit supremum of a real sequence (x_n) is the extended real number,

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right)$$

Similarly, the limit infimum of a real sequence (x_n) is the extended real number,

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right)$$

Below are a few (unproven) theorems regarding limits infimum/supremum.

Theorem 2 (Subsequence Characterization of Limsup/Liminf)

If (x_n) is a real sequence, $s := \limsup_{n\to\infty} x_n$ and $t := \liminf_{n\to\infty} x_n$ then there are subsequences (x_{n_k}) and (x_{n_ℓ}) of (x_n) such that $x_{n_k} \to s$ as $k \to \infty$ and $x_{n_\ell} \to t$ as $\ell \to \infty$.

Theorem 3 (Convergence Criterion of Limsup/Liminf)

Let $(x_n) \in \mathbb{R}^{\mathbb{N}}$ and $x \in \mathbb{R}^*$. Then $x_n \to x$ as $n \to \infty$ if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = x$.

Theorem 4

If $(x_n) \in \mathbb{R}^{\mathbb{N}}$ then $\limsup_{n\to\infty} x_n \geq \liminf_{n\to\infty} x_n$.

Theorem 5

A sequence $(x_n) \in \mathbb{R}^{\mathbb{N}}$ is bounded above if and only if $\limsup_{n\to\infty} x_n < \infty$.

Theorem 6 (Monotonicity of the Limit Supremum)

If $x_n < y_n$ for large n then $\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n$.