3. Infima and Suprema

In this topic, we will discuss various concepts related to infima and suprema, which build on the idea of minima and maxima from calculus.

Definition 1 (Bound and Boundedness)

We say that some subset $E \subset \mathbb{R}$ is bounded if, for all $e \in E$, $|e| < M$ for some $M \in \mathbb{R}$. We call M the bound.

Through the following example, we observe that a bounded sequence has multiple bounds.

Example 1

The set $\{1, 2, 3\}$ is bounded. One possible upper bound is 3. Another possible upper bound is 4. You may notice, there are infinitely many bounds for a bounded set.

We shall now formalize the notion of the "best" bound.

Definition 2 (Infimum and Supremum)

The infimum of some set E is the greatest lower bound. That is, it is the greatest $M \in \mathbb{R}$ such that for all $e \in E$, $e \geq M$.
The supremum of some set E is the least upper bound. That is, it is the least $M \in \mathbb{R}$ such that for all $e \in E$, $e \leq M$.

You should observe that it is possible for the infimum/supremum of a set to be in the set, but it is also possible for it to be outside the set.

Example 2

Consider set $E := \{x^{-2} : x \in \mathbb{R}_{>0}\} \subset \mathbb{R}$. Here, the supremum of E is undefined, since as x tends towards 0, the values in the set tend toward $+\infty$. Also, $\inf E = 0$ as 0 is the greatest lower bound.

We shall take the following property axiomatically – that is, we will assume it without proof.

Theorem 1 (Completeness Axiom)

If E is a nonempty subset of $\mathbb{R}$ that is bounded above then E has a finite supremum.

At this time, we shall explore several key results involving boundedness and infima/suprema.
We shall now explore a fundamental, ancient result – the Archimedean Principle. Although the groundwork for the principle was indeed laid by the ancient Greek Archimedes, he did not formalize the principle. Instead, this was done by Eudoxus, a subsequent Greek mathematician.

Theorem 1 (Archimedean Principle)

For all $a, b \in \mathbb{R}_{>0}$, there exists an $n \in \mathbb{N}$ such that $b < na$.

Proof. Let $a, b \in \mathbb{R}$. Consider the set $E := \{na : n \in \mathbb{N}\}$, i.e., the set of all natural number multiples of a. Observe that E is not bounded above. Suppose, for sake of contradiction, that there exists no $n \in \mathbb{N}$ such that $b < na$. Then b would be an upper bound for E, which contradicts the fact that E is not bounded above. Thus, there must exist some $n \in \mathbb{N}$ to satisfy the inequality. □

Observe that if $b < a$ already, then $n = 1$ satisfies this inequality. However, if $b > a$ then we must find some $n > 1$ to satisfy the inequality; the Archimedean principle dictates that we can always find such a natural number.

Theorem 2 (Approximation Property for Suprema)

Let E be some set with finite supremum. Let $\varepsilon > 0$. Then there exists some $e \in E$ such that $\sup E - \varepsilon < e \leq \sup E$.

Theorem 3 (Monotone Property)

Let A and B be nonempty subsets of $\mathbb{R}$ such that $A \subset B$. If B has a finite supremum then $\sup A \leq \sup B$.

Proof. Suppose B has a finite supremum, specifically let $s_B := \sup B$. Then by the Approximation Property for Suprema, for every $\varepsilon > 0$ there exists some $b \in B$ such that $s - \varepsilon < b \leq s$. Observe that s_B is also an upper bound for A. Since A is nonempty and bounded above, by Completeness Axiom it has a finite supremum, $s_A := \sup A$. Suppose for sake of contradiction that $s_A > s_B$. By the Approximation Property for Suprema, for any $\varepsilon > 0$ there exists some $a \in A$ such that $s_a - \varepsilon < a \leq s_A$. Choose $\varepsilon = \frac{s_A - s_B}{2} > 0$. Then we get the inequality $s_A - \frac{s_A - s_B}{2} < a \leq s_A$. Simplification yields $\frac{s_A + s_B}{2} < a \leq s_A$. Since $a \in A$ and $A \subset B$ we know $a \in B$ but then $a > \frac{s_A + s_B}{2} > s_B$. This contradicts the fact that s_B is an upper bound for B. □

So far, our theorems have been specific to suprema. The following result allows results on suprema to be applied to infima.

Theorem 4 (Reflection Principle)

For $E \subset \mathbb{R}$ where E is nonempty, E has a supremum iff $-E$ has an infimum. Specifically, $-\sup E = \inf -E$.

Proof. Assume $s := \sup E$ exists. For any $y \in -E$, $y = -x$ for some $x \in E$. Since $x \leq s$, we have $-x \geq -s$. Thus $-s$ is a lower bound for $-E$. Let m be any lower bound for $-E$. Then $-m$ is an upper bound for E, so $s \leq -m$ or equivalently $m \leq -s$. Therefore $-s = \inf -E$. □

We have one final useful definition:

Definition 3 (Extended Real Number)

The extended real numbers $\mathbb{R}^*$ is defined as $\mathbb{R} \cup \{\infty\}$ where $\infty^{-1} = 0, 0^{-1} = \infty$ and $\infty \cdot a = \infty$ for all $a \in \mathbb{R}$.