2. Metrics, Absolute Value and Topology of $\mathbb{R}$

Now, we shall define a few extra notational tools that we can use.

Definition 1 (Metric)

A metric (or distance) is a function $d : E \times E \to \mathbb{R}$ which satisfies the following properties:

(a) d is nonnegative, that is, for all $u, v \in E$, $d(u, v) \geq 0$ and $d(u, v) = 0$ if and only if $u = v$.

(b) d is commutative, that is, for all $u, v \in E$, $d(u, v) = d(v, u)$.

(c) d must satisfy the triangle inequality, that is, for all $u, v, w \in E$, $d(u, v) \leq d(u, v) + d(v, w)$.

In plain language, a distance must never be negative and can only be zero if both inputs are identical. The distance between u and v must be the same as the distance between v and u. Finally, it must always be less distance to go directly from u to v, rather than to add an intermediary w.

Definition 2 (Metric Space)

A metric space is a pair (E, d) where E is a set and d is a metric. We typically call the elements of a metric space points.

Example 1 (Examples of Metrics)

- The Euclidean distance on $\mathbb{R}^2$, $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, is defined by $d : (u, v) \mapsto \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.
- The Manhattan distance on $\mathbb{R}^2$, $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, is defined by $d : (u, v) \mapsto |x_1 - x_2| + |y_1 - y_2|$.
- The Hamming distance on $\mathbb{Z}_2^n$, $d : \mathbb{Z}_2^n \times \mathbb{Z}_2^n \to \mathbb{R}$, is sends (u, v) to the number of $i \in \{1, ..., n\}$ where $u_i \neq v_i$. The Hamming distance is used in error detection/correction when transmitting digital signals.

Definition 3 (Absolute Value)

The absolute value of $a \in \mathbb{R}$ is,

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Notice that $|\cdot|$ defines a norm.

Definition 4 (Metric Defined by $|\cdot|$)

Assume that $|\cdot|$ is a norm on some set E. Then we say that the metric defined by $|\cdot|$ on E is $d : E \times E \to \mathbb{R}$ such that $d : (u, v) \mapsto |u - v|$.

Note that the ℓ^2 norm in $\mathbb{R}^n$ defines the Euclidean distance in $\mathbb{R}^n$ from linear algebra. Similarly, the ℓ^1 norm in $\mathbb{R}^n$ defines the Manhattan distance.

Observe that, for any $x, y \in \mathbb{R}$, the statement $|x| < y$ is equivalent to $-y < x < y$. Be careful of the inequality here! However, this statement may prove beneficial in proofs.

Now we shall introduce notation of intervals and their higher dimensional analogue – the ball.

Definition 5 (Intervals)

(a) A closed interval $[a, b]$ is defined as $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$.

(b) An open interval (a, b) is defined as $(a, b) := \{x \in \mathbb{R} : a < x < b\}$.

Definition 6 (Ball)

A ball in some metric space (E, d) of radius $R > 0$ centered at x_0 is defined to be

$$B_R(x_0) := \{x \in E : d(x, x_0) < R\}$$

Observe that the ball over the metric space $\mathbb{R}$ with the metric induced by $|\cdot|$ is an open interval.