LAST TIME, WE SPOKE ABOUT CONCEPTS LEADING UP TO THE BASIS.
REMEMBER $\operatorname{span}\{v_1, v_2, \dots, v_n\} = \{\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n : \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}\}$.
AND $\{v_1, v_2, \dots, v_n\}$ IS L.I. iff $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0 \iff \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$
AND A BASIS FOR V IS A SET $B = \{v_1, v_2, \dots, v_n\}$ SO THAT $\operatorname{span} B = V$ AND B IS L.I.

TODAY WE WILL DEFINE A FEW VECTOR SPACES ASSOCIATED WITH A MATRIX A.

(I) THE <u>COLUMNSPACE</u> OF A IS THE SPAN OF THE SET OF COLUMN VECTORS OF A.

eg/

$$A = \begin{bmatrix} 2 & 7 & 5 & 2 \\ 1 & 4 & 8 & 6 \\ 6 & 5 & 9 & 1 \end{bmatrix}$$

$$\text{columnspace}(A) = \operatorname{span}\left\{\begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}\right\}$$

NOTE: THIS IS A SPAN, <u>NOT</u> A BASIS.

(II) THE <u>ROWSPACE</u> OF A IS THE SPAN OF THE SET OF ROW VECTORS OF A.

eg/

$$A = \begin{bmatrix} 2 & 7 & 5 & 2 \\ 1 & 4 & 8 & 6 \\ 6 & 5 & 9 & 1 \end{bmatrix}$$

$$\text{rowspace}(A) = \operatorname{span}\left\{\begin{bmatrix} 2 \\ 7 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 8 \\ 6 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 9 \\ 1 \end{bmatrix}\right\}$$

AGAIN, THIS IS A SPAN, <u>NOT</u> A BASIS

(III) THE <u>NULLSPACE</u> OF A IS THE SET OF VECTORS x WHICH SOLVE $Ax = 0$.

RECALL: THE HOMOGENEOUS SYSTEM $Ax = 0$ HAS AT LEAST THE TRIVIAL SOLUTION.

SO $\text{nullspace}(A) = \{x \in \mathbb{R}^n : Ax = 0\}$

FROM THESE, WE DEFINE 4 FUNDAMENTAL SPACES OF A MATRIX:

I. columnspace(A)

II. rowspace(A)

III. nullspace(A)

IV. nullspace(A^T)

EACH OF THESE IS A VECTOR SPACE.

NOTE: $\text{columnspace}(A^T) = \text{rowspace}(A)$ AND $\text{rowspace}(A^T) = \text{columnspace}(A)$

RECALL LAST TIME, I MENTIONED THAT THE SPANNING SET "PLATEAUS" ONCE IT EQUALS THE WHOLE VECTOR SPACE, AND AFTER A CERTAIN MAXIMUM NUMBER OF VECTORS, A SET OF VECTORS WILL ALWAYS BE L.D. THE "MAGIC NUMBER" OF VECTORS WHERE BOTH OF THESE HAPPEN IS THE <u>DIMENSION</u> OF THE VECTOR SPACE. THIS IS ALSO THE # OF VECTORS IN A BASIS.

eg, CONSIDER THE SET $V = \left\{ \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\} \subseteq \mathbb{R}^3$. THIS IS A VECTOR SPACE (CAN USE THE 2-STEP SUBSPACE TEST.)

A BASIS FOR V IS $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$. THIS BASIS HAS 1 ELEMENT SO $\dim V = 1$.

NOTE: ALL BASES FOR A VECTOR SPACE WILL HAVE THE SAME # OF VECTORS, AND SO A VECTOR SPACE WILL HAVE A SINGLE, SPECIFIC DIMENSION.

<u>PROPERTIES OF DIMENSION AND SUBSPACES</u>

SUPPOSE V IS A VECTOR SPACE AND W IS A SUBSPACE OF V. THEN,

(1) $\dim W \leq \dim V$

(2) $\dim W = \dim V$ IFF $W = V$

SOME COMMON DIMENSIONS:

- $\dim \mathbb{P}_n = n+1$ WHERE $\mathbb{P}_n$ DENOTES POLYNOMIALS OF DEGREE AT MOST n
- $\dim \mathbb{R}^n = n$
- $\dim M_{n \times m} = nm$ WHERE $M_{n \times m}$ IS $n \times m$ MATRICES

WE DEFINE THE <u>RANK</u> OF A MATRIX AS THE DIMENSION OF THE COLUMNSPACE OR THE DIMENSION OF THE ROWSPACE. (N.B., THESE TWO DIMENSIONS ARE ALWAYS EQUAL.)

WE DEFINE THE <u>NULLITY</u> OF A MATRIX AS THE DIMENSION OF THE NULLSPACE.

<u>RANK-NULLITY THM. FOR MATRICES.</u>

SUPPOSE A IS AN $m \times n$ MATRIX. THEN,

$$\text{nullity}(A) + \text{rank}(A) = n$$

<u>ORTHOGONAL COMPLEMENT</u>

IF WE HAVE A VECTOR SPACE V WITH W A SUBSPACE OF V, THEN DEFINE THE <u>ORTHOGONAL COMPLEMENT</u> OF W AS $W^\perp = \{ v \in V : v \cdot w = 0 \ \forall w \in W \}$, THAT IS, ANY VECTOR v IN $W^\perp$ IS ORTHOGONAL TO <u>EVERY</u> VECTOR IN W.

NOTE: IF W IS A SUBSPACE OF V THEN $W^\perp$ IS A SUBSPACE OF V.