LAST WEEK, WE SPOKE ABOUT MATRICES AS TRANSFORMS.
IF T_A AND T_B ARE TRANSFORMATION MATRICES, THEN $T_B \circ T_A$ IS THE COMPOSITION
(BE VERY MINDFUL OF THE ORDER — THIS IS T_A APPLIED FIRST, THEN T_B.)
WE CAN CALCULATE THE MATRIX BY $[T_B][T_A] = [T_B \circ T_A]$

NOW A MAP IS CALLED ONE-TO-ONE IF IT MAPS DISTINCT THINGS IN THE DOMAIN TO DISTINCT THINGS IN THE CODOMAIN.
A MAP IS CALLED ONTO IF THE CODOMAIN IS THE RANGE.
THE KERNEL OF A MAP IS THE SET OF ALL THINGS MAPPED TO ZERO.
eg, CONSIDER THE MAP $f(x) = \sin(x)$ WHERE $f: \mathbb{R} \to \mathbb{R}$.
HERE $\text{range}(f) = [-1, 1]$ AND $\ker(f) = \{k\pi : k \in \mathbb{Z}\}$

WE CAN MAKE A "TRANSFORM ANALOG" OF A MATRIX INVERSE. THIS IS A TRANSFORM ASSOCIATED WITH THE INVERSE MATRIX. WHEN COMPOSED, WE GET THE IDENTITY TRANSFORM.

NOW, LET'S SHIFT A BIT AND TALK ABOUT EIGENVECTORS!
AN EIGENVECTOR IS A VECTOR FOR WHICH APPLYING SOME TRANSFORMATION ONLY STRETCHES OR SHRINKS THE VECTOR, BUT DOES NOT CHANGE ITS DIRECTION.
THE EIGENVALUE IS THE AMOUNT THE EIGENVECTOR IS SCALED BY.
TO FIND THE EIGENPAIR, WE NEED TO SOLVE $A\xi = \lambda \xi$.

① TO FIND EIGENVALUES, REARRANGE $A\xi = \lambda\xi$ TO GET:

$$A\xi - \lambda\xi = 0$$
$$\Rightarrow (A - I\lambda)\xi = 0$$
$$\Rightarrow \det(A - I\lambda) = 0$$

THIS WILL YIELD THE CHARACTERISTIC POLYNOMIAL WHICH IS A POLYNOMIAL WITH VARIABLE λ. SOLVE IT TO FIND VALUE(S) FOR λ.

② TO FIND EIGENVECTORS, GO THROUGH EACH EIGENVALUE AND SOLVE FOR:

$$(A - I\lambda)\xi = 0$$

THIS GIVES YOU ONE EIGENVECTOR FOR EACH EIGENVALUE.

eg/ FIND EIGENPAIRS FOR $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.

① $A - I\lambda = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\lambda = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}$

SO $\det(A - I\lambda) = (-\lambda)(-3-\lambda) + 2(1) = \lambda^2 + 3\lambda + 2 = (\lambda+2)(\lambda+1)$

SO $\lambda_1 = -1$, $\lambda_2 = -2$

② FOR $\lambda_1 = -1$, WE FIND EIGENVECTOR $\vec{\xi}^{(1)}$:

$(A - I\lambda)\vec{\xi}^{(1)} = \vec{0} \Rightarrow \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -\xi_1\lambda + \xi_2 \\ -2\xi_1 - 3\xi_2 - \lambda\xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

SO SOLVE SYSTEM $\begin{cases} -\xi_1\lambda + \xi_2 = 0 \\ -2\xi_1 - (3+\lambda)\xi_2 = 0 \end{cases}$ WITH $\lambda = -1$

$\Rightarrow \begin{cases} \xi_1 + \xi_2 = 0 \\ -2\xi_1 - 2\xi_2 = 0 \end{cases} \Rightarrow \xi_1 = -\xi_2$ (FROM TOP)

(NOTE: YOU CAN PICK EITHER TOP OR BOTTOM $\Rightarrow$ DO WHICHEVER IS EASIER AND SET ONE TO 1.)

SO $\xi^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

FIND $\xi^{(2)}$ WITH $\lambda_2 = -2$: $\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 2\xi_1 + \xi_2 = 0 \Rightarrow \xi_2 = -2\xi_1$

SO $\xi^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

FINALLY, WE SAY $\text{nullspace}(A - I\lambda)$ IS THE EIGENSPACE OF A WITH RESPECT TO EIGENVALUE λ.