1. Linear Systems and Gaussian Elimination

Welcome to linear algebra! I hope you enjoy this course.

You might remember solving linear systems of equations in high school. If you do, you'll certainly remember that there was never a single clear way to solve systems. Easy systems felt natural to solve, but not all systems could be so easily solved. This is a fundamental idea that we'll be dealing with as the course progresses. Let's start with a definition:

Definition 1 (Linear Equation)

An equation is called linear if each variable has no "special operations" such as exponents, roots, logarithms, or trigonometric functions (e.g., $\sin$).

When a question asks to "solve a system of linear equations", it is really asking you to find the place(s) where all of the listed equations intersect. The simplest case of this is two, 2-dimensional lines. We remember how to find the point of intersection of two lines from high school. A line extended to three dimensions is a plane.

You might take relief in learning that any system of linear solutions can be clearly classified based on the number of solutions it has. It may have either zero, one, or infinitely-many solutions. So there is no possibility of having 2 solutions, 3 solutions, or any other number of solutions.

Definition 2 (Consistent)

We say that a system is consistent if it has at least one solution. That is, a system is consistent if it has one or infinitely-many solutions, and is inconsistent only if it has zero solutions.

Example 1

Solve the following system of linear equations:

$$\begin{cases} x - y &= 1 \\ 2x + y &= 6 \end{cases}$$

We can use simple high school math to solve this problem. First, we rearrange the first equation to get $x = y + 1$. Then we can substitute this into the second equation for x to yield $2(y+1) + y = 6$ which, after rearranging, yields $y = \frac{4}{3}$. Substituting this back into one of the two equations (say, $x - y = 1$), we get $x - \frac{4}{3} = 1$ and so $x = \frac{7}{3}$. Hence the solution to our system is $(x, y) = (\frac{7}{3}, \frac{4}{3})$.

This was a relatively simple example. However, adding more variables or equations can increase the complexity substantially. We can turn any system into a matrix, which is a "grid" of numbers.

Example 2

Solve the following system of linear equations:

$$\begin{cases} x + y + 2z = 9 \\ 2x + 4y - 3z = 1 \\ 3x + 6y - 5z = 0 \end{cases} \implies \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

We call this matrix the augmented matrix. Note that the line has no mathematical implication. It just serves as a reminder that the left side is variable coefficients and the right side is constants. When doing operations on the matrix, the line can be ignored.

Definition 3 (Elementary Row Operation)

Elementary row operations are operations or changes that can be made to a matrix, while maintaining the same solutions. There are three such operations which do change the matrix but do not change the solutions:

1. Multiply everything in a row by a number (except 0).
2. Swap two rows.
3. Add a constant times one row to another.

Keep in mind that you should only apply one operation at a time. Further, you should be very clear about which operations you are using. You should also note that the two matrices (before and after an operation is applied) are not equal.

Definition 4 (Homogeneous System)

The homogeneous system has all zeros for the constants.

We can solve a system using an elimination process. First, we should discuss some forms for matrices.

Definition 5 (Row-Echelon Form)

If a matrix is in row-echelon form (REF), it must meet the following conditions:

1. If a row is not all zeros, the first non-zero number is a 1. This is called a leading one.
2. Any rows that have all zeros are at the bottom of the matrix.
3. The leading ones form a "staircase" shape.

Definition 6 (Reduced Row-Echelon Form)

If a matrix is in reduced row-echelon form (RREF), it must meet all three conditions of row-echelon form and must meet the condition that every column with a leading one has zeros everywhere else.

There are two types of elimination:

Definition 7 (Gaussian Elimination)

Gaussian elimination is the application of elementary row operations until the matrix is in REF.

Definition 8 (Gauss-Jordan Elimination)

Gauss-Jordan elimination is the application of elementary row operations until the matrix is in RREF.

Now we will examine an example from earlier.

Example 3

Solve the following system of linear equations:

$$\begin{cases} 3x + y - z = 1 \\ x - y + z = 3 \\ 2x + y + z = 0 \end{cases} \implies \left[\begin{array}{ccc|c} 3 & 1 & -1 & 1 \\ 1 & -1 & 1 & 3 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 1 & -1 & 1 \\ 1 & -1 & 1 & 3 \\ 2 & 1 & 1 & 0 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 3 & 1 & -1 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right] \overset{R_2 : R_2 - 3R_1}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 4 & -4 & -8 \\ 2 & 1 & 1 & 0 \end{array}\right] \overset{R_3 : R_3 - 2R_1}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 4 & -4 & -8 \\ 0 & 3 & 1 & -6 \end{array}\right]$$

$$\overset{R_2 : -\frac{1}{4} \times R_2}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 3 & 1 & -6 \end{array}\right] \overset{R_3 : R_3 - 3R_2}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & 0 \end{array}\right] \overset{R_3 : \frac{1}{2} \times R_3}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Now this matrix is in REF but not RREF. We will use Gauss-Jordan elimination and continue to row reduce to RREF.

$$\overset{R_2 : R_2 + R_3}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right] \overset{R_1 : R_1 - R_3}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right] \overset{R_1 : R_1 + R_2}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Now that we are in RREF, we can just read off the solution: $(x, y, z) = (1, -2, 0)$.