6. Change of Basis and Commutative Diagrams

This week, we will discuss change of basis. Let's break this down into three topics:

I. Writing a coordinate vector for a given basis.

Definition 1 (Coordinate Vector)

Suppose $v \in V$ and $B = \{b_1, ..., b_n\}$ is a basis for V. Then if $v = \alpha_1 b_1 + ... + \alpha_n b_n$, we say that the coordinate vector for v under the basis B is $[v]_B = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$

Example 1

Say $v = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ and $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. Then $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. This implies that $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ 0 \\ \alpha_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha_2 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} \alpha_3 \\ \alpha_3 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_3 \\ \alpha_2 + \alpha_3 \\ \alpha_1 + \alpha_2 \end{bmatrix}$. This yields a system of equations:

$$\begin{cases} \alpha_1 + \alpha_3 = 3 \\ \alpha_2 + \alpha_3 = 2 \\ \alpha_1 + \alpha_2 = 1 \end{cases}$$

This system can be solved using Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{array}\right] \overset{RREF}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

So we write that $[v]_B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

II. Finding $[I]_{B \leftarrow A}$.

Definition 2 (Change of Basis Identity Matrix)

Let $A = \{a_1, ..., a_n\}$ and $B = \{b_1, ..., b_n\}$ be bases for some vector space V. We define the change of basis identity matrix $[I]_{B \leftarrow A}$ as the linear transformation which, for any vector v, $[I]_{B \leftarrow A}[v]_A = [v]_B$. It is computed as $[I]_{B \leftarrow A} = \begin{bmatrix} [b_1]_A & \cdots & [b_n]_A \end{bmatrix}$

Example 2

Say $A = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \right\}$ and $B = \left\{ \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} \right\}$ which are both bases for $\mathbb{M}_{2\times 2}$.
Write each the coordinate vector for each $b \in B$ under the basis A:

$$\left[\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}\right]_A = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \left[\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\right]_A = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \left[\begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix}\right]_A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \left[\begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}\right]_A = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

Placing these in as columns for our matrix $[I]_{B\leftarrow A}$ yields,

$$[I]_{B\leftarrow A} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

III. Finding $[T]_{B\leftarrow A}$.

Example 3

Let V and W be vector spaces. Say $T : V \to W$ where A is a basis of V and B is a basis of W. Let S_V and S_W be the standard bases for V and W. We can use a commutative diagram to find $[T]_{B\leftarrow A}$ in a few steps:

$$\begin{array}{ccc} A & \xrightarrow{[T]_{B\leftarrow A}} & B \\ {\scriptstyle [I]_{S_V\leftarrow A}}\downarrow & & \uparrow{\scriptstyle [I]_{B\leftarrow S_W}} \\ S & \xrightarrow[{[T]=[T]_{S_W\leftarrow S_V}}]{} & S \end{array}$$

The idea of a commutative diagram is that if there are two paths between two nodes, the two paths are equivalent. In this context, this means that $[T]_{B\leftarrow A} = [I]_{B\leftarrow S_W}[T]_{S_W\leftarrow S_V}[I]_{S_V\leftarrow A}$.
We would find $[I]_{B\leftarrow S_W}$ and $[I]_{S_V\leftarrow A}$ as we did in (II). We find $[T]_{S_W\leftarrow S_V}$ as we did a while back.