02 - Some Special Functions

At this point, we will introduce a few new functions, which we will start to incorporate into future integrands! It may come as a surprise, but there are plenty of non-elementary integrands.

Definition 1 (Dirac Delta Function)

The Dirac delta function is defined as,

$$\delta(x) := \begin{cases} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{cases}$$

such that,

$$\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1$$

One useful property of the Dirac delta function is the **sifting property**:

Theorem 1 (Sifting Property of Dirac Delta)

Let $c \in (a, b)$ and let $f(x)$ be any sufficiently smooth function. Then,

$$\int_a^b f(x)\delta(x-c)\,\mathrm{d}x = f(c)$$

Example 1

Evaluate the following integrals.

$$\int_{-3}^{2} (x^2 - 2x + 1)\delta(x-1)\,\mathrm{d}x$$

$$\int_0^{\pi} \sin(x)\delta\left(x - \frac{\pi}{2}\right)\mathrm{d}x$$

$$\int_{-1}^{1} e^{x^2}\delta\left(x + \frac{1}{2}\right)\mathrm{d}x$$

$$\int_{-1}^{1} e^{x^2}\delta(x+2)\,\mathrm{d}x$$

Definition 2 (Heaviside Step Function)

The Heaviside step function is defined as,

$$\mathscr{U}(x) := \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Keen students will notice that $\displaystyle\int_{-\infty}^{x} \delta(t)\,\mathrm{d}t = \mathscr{U}(x)$, or equivalently $\dfrac{\mathrm{d}}{\mathrm{d}x}\mathscr{U}(x) = \delta(x)$. This begs the question—what is the antiderivative of the Heaviside step function?

Example 2

Evaluate the following integral:

$$\int_{-\infty}^{x} \mathscr{U}(t)\,\mathrm{d}t$$

We shall now see how a product of Heaviside step functions creates a "window".

Example 3

Evaluate the following definite integral:

$$\int_{-\pi}^{\pi} \sin(x)\mathscr{U}(x)\mathscr{U}\left(\frac{\pi}{6} - x\right)\mathrm{d}x$$

We will now move on to slightly more complex (yet perhaps more approachable) functions. First is the product logarithm.

Definition 3 (Product Logarithm)

Let the product logarithm $y = W(x)$ be the solution to the equation,

$$ye^y = x$$

We shall start by finding the derivative and the antiderivative of $W(x)$.

Example 4

Find $\dfrac{\mathrm{d}}{\mathrm{d}x}W(x)$ and $\displaystyle\int_{-\infty}^{x} W(t)\,\mathrm{d}t$.

Example 5

Evaluate the following integral:

$$\int_0^{-e^{-1}} \frac{x}{W(x)}\,\mathrm{d}x$$

The final special function we will explore is the error function.

Definition 4 (Error Function)

We define the error function erf as,

$$\operatorname{erf} x := \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \, \mathrm{d}t$$

You may remember from a previous calculus course that this function is "not integrable". What your instructor meant by this is that the integral was not expressible in terms of the functions you know. We, instead, define the answer to that integral to be the error function. Then we can do more integrals with it!

Example 6

Evaluate the following antiderivative:

$$\int_{-\infty}^{x} \operatorname{erf} t \, \mathrm{d}t$$