EARLIER, WE LEARNED ABOUT THE DOT PRODUCT. THIS ONLY APPLIED TO EUCLIDEAN SPACES, $\mathbb{R}^n$. BUT CAN WE THINK ABOUT A SIMILAR PRODUCT FOR GENERAL VECTOR SPACES? NOTE: THIS IS <u>NOT</u> SCALAR MULTIPLICATION, WHICH IS A PRODUCT BETWEEN A VECTOR AND A SCALAR. THE DOT PRODUCT TAKES TWO VECTORS!

THE GENERAL VERSION IS CALLED AN <u>INNER PRODUCT</u>, DENOTED AS $\langle u, v \rangle$.

AN INNER PRODUCT MUST SATISFY THESE 4 AXIOMS:

1. $\langle u, v \rangle = \langle v, u \rangle$ (SYMMETRY)
2. $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ (ADDITIVITY)
3. $\langle ku, v \rangle = k \langle u, v \rangle$ (HOMOGENEITY)
4. $\langle v, v \rangle \geq 0$ AND $\langle v, v \rangle = 0$ IFF $\bar{v} = \bar{0}$ (NONNEGATIVITY)

A VECTOR SPACE ON WHICH AN INNER PRODUCT IS DEFINED IS CALLED AN <u>INNER PRODUCT SPACE</u>

A COUPLE EXAMPLES OF INNER PRODUCTS:

I. IN $\mathbb{R}^n$, $\langle u, v \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$ (DOT PRODUCT)

II. IN $\mathbb{R}^n$, $\langle u, v \rangle_w = w_1 u_1 v_1 + w_2 u_2 v_2 + \cdots + w_n u_n v_n$ (WEIGHTED DOT PRODUCT)

III. IN C^∞, $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) g(x) dx$ (L^2 INNER PRODUCT)

FROM THIS, WE CAN DEFINE/EXTEND A FEW MORE CONCEPTS

THE <u>NORM</u> OR <u>LENGTH</u> OR <u>MAGNITUDE</u> OF A VECTOR $v \in V$, DENOTED $\|v\|$ IS $\|v\| = \sqrt{\langle v, v \rangle}$.

THE <u>DISTANCE</u> BETWEEN TWO VECTORS $u, v \in V$ IS $d(u,v) = \|u - v\|$.

JUST LIKE IN $\mathbb{R}^n$, TWO VECTORS ARE ORTHOGONAL IF $\langle u, v \rangle = 0$

WE SAY A SET OF VECTORS IS ORTHOGONAL IF EACH PAIR OF VECTORS IS ORTHOGONAL.

AN ORTHOGONAL SET WHERE EACH VECTOR HAS $\|v\| = 1$ IS CALLED ORTHONORMAL.

FROM EARLIER, RECALL THAT THERE ARE MANY POSSIBLE BASES FOR A VECTOR SPACE, ALL EQUALLY CORRECT. A SPECIAL BASIS IS CALLED AN ORTHONORMAL BASIS, WHERE EVERY BASIS VECTOR IS ORTHONORMAL.

EVERY NONZERO, FINITE-DIMENSIONAL INNER PRODUCT SPACE HAS AN ORTHONORMAL BASIS.

TO COME UP WITH AN ORTHONORMAL BASIS, WE USE THE GRAM-SCHMIDT PROCESS. FIRST, FIND A BASIS (NOT NECESSARILY ORTHONORMAL). FOR EXAMPLE, SAY $\{u_1, u_2, \ldots, u_n\}$ IS A BASIS.

STEP 1: SET $v_1 = u_1$

STEP 2: SET $v_2 = u_2 - \text{proj}_{W_1} u_2$ WHERE $W_1 = \text{span}\{v_1\}$

STEP 3: SET $v_3 = u_3 - \text{proj}_{W_2} u_3$ WHERE $W_2 = \text{span}\{v_1, v_2\}$

$\vdots$

STEP n: SET $v_n = u_n - \text{proj}_{W_{n-1}} u_n$ WHERE $W_{n-1} = \text{span}\{v_1, v_2, \ldots, v_{n-1}\}$

ONCE YOU FINISH THESE STEPS, $\{v_1, v_2, \ldots, v_n\}$ WILL BE AN ORTHOGONAL BASIS, BUT NOT NECESSARILY ORTHONORMAL. NOW, WE NEED TO NORMALIZE, SO $\hat{v}_i = \frac{v_i}{\|v_i\|}$. THIS WILL GUARANTEE THAT $\|\hat{v}_i\| = 1$, SO $\{\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n\}$ IS AN ORTHONORMAL BASIS.

THE PROJECTIONS ARE COMPUTED BY: $\text{proj}_{W_{i-1}} u_i = \langle u_i, v_1 \rangle v_1 + \langle u_i, v_2 \rangle v_2 + \cdots + \langle u_i, v_{i-1} \rangle v_{i-1}$.