YOU'VE PROBABLY SOLVED LINEAR SYSTEMS BEFORE (IN HIGH SCHOOL), BUT IT WAS ALWAYS A "DIFFICULT" PROCESS.

ANY SYSTEM OF LINEAR EQUATIONS HAS 3 POSSIBILITIES:

- NO SOLUTIONS
- ONE SOLUTION
- INFINITELY-MANY SOLUTIONS.

WE CALL A SYSTEM <u>CONSISTENT</u> IF IT HAS AT LEAST 1 SOLUTION.

eg/ SOLVE THE SYSTEM:

$$\begin{cases} x - y = 1 & \Rightarrow x = 1 + y \\ 2x + y = 6 & \Rightarrow 2(1+y) + y = 6 \Rightarrow 2 + 2y + y = 6 \Rightarrow 3y = 4 \Rightarrow y = \frac{4}{3} \Rightarrow x = 1 + \frac{4}{3} = \frac{7}{3} \end{cases}$$

BUT THINGS GET HARDER IF WE HAVE 3 EQUATIONS OR AN EXTRA VARIABLE.

WE CAN TURN ANY SYSTEM INTO A <u>MATRIX</u>, WHICH IS LIKE A "GRID" OF NUMBERS.

eg/

$$\begin{cases} x + y + 2z = 9 \\ 2x + 4y - 3z = 1 \\ 3x + 6y - 5z = 0 \end{cases} \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

COEFFICIENTS OF THE VARIABLES

CONSTANTS

LIKE THE "=" SIGN

WE CALL THIS MATRIX THE <u>AUGMENTED MATRIX</u>.

NOTE: THE LINE HAS <u>NO</u> MATHEMATICAL IMPLICATION, IT JUST SERVES AS A LITTLE REMINDER THAT THE LEFT SIDE IS VARIABLE COEFFICIENTS AND THE RIGHT SIDE IS CONSTANTS.

<u>ELEMENTARY ROW OPERATIONS</u> CAN BE PERFORMED WITHOUT CHANGING FINAL SOLUTION:

1) MULTIPLY EVERYTHING IN A ROW BY A NUMBER (EXCEPT 0).
2) SWAP 2 ROWS
3) ADD A CONSTANT TIMES ONE ROW TO ANOTHER.

THE <u>HOMOGENEOUS SYSTEM</u> HAS ALL ZEROS FOR THE CONSTANTS.

WE CAN SOLVE A SYSTEM USING GAUSSIAN ELIMINATION:

eg/ SOLVE THIS SYSTEM:

$$\begin{cases} 3x + y - z = 1 \\ x - y + z = 3 \\ 2x + y + z = 0 \end{cases} \xrightarrow[\text{MATRIX}]{\text{AUGMENTED}} \left[\begin{array}{ccc|c} 3 & 1 & -1 & 1 \\ 1 & -1 & 1 & 3 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

LET'S FIRST TALK ABOUT ROW-ECHELON FORM AND REDUCED ROW-ECHELON FORM.

<u>ROW-ECHELON FORM</u> MUST MEET THESE PROPERTIES:

1) IF A ROW ISN'T ALL ZEROS, THE FIRST NON-ZERO NUMBER IS A 1. THIS IS CALLED A <u>LEADING ONE</u>.
2) ANY ROWS THAT HAVE ALL ZEROS ARE AT THE BOTTOM OF THE MATRIX.
3) THE LEADING ONES FORM A "STAIRCASE" SHAPE.

<u>REDUCED ROW-ECHELON FORM (RREF)</u> ALSO MUST MEET

4) EVERY COLUMN WITH A LEADING 1 HAS ZEROS EVERYWHERE ELSE.

NOW, LET'S RETURN TO THE EXAMPLE.

eg/

$$\left[\begin{array}{ccc|c} 3 & 1 & -1 & 1 \\ 1 & -1 & 1 & 3 \\ 2 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 3 & 1 & -1 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 : R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 4 & -4 & -8 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 : R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 4 & -4 & -8 \\ 0 & 3 & 1 & -6 \end{array}\right] \xrightarrow{R_2 : R_2/4} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 3 & 1 & -6 \end{array}\right] \xrightarrow{R_3 : R_3 - 3R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 : R_3/2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

NOW THIS MATRIX IS IN ROW-ECHELON FORM (REF) BUT NOT RREF.

<u>GAUSSIAN ELIMINATION</u> — STOPS AT REF

<u>GAUSS-JORDAN ELIMINATION</u> — GOES ALL THE WAY TO RREF.

LET'S PROCEED WITH GAUSS-JORDAN:

$$\xrightarrow{R_2 : R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 : R_1 - R_3} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 : R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

SO NOW YOU HAVE 1 ALONG DIAGONAL, 0 EVERYWHERE ELSE. IT IS IN RREF. EVERYTHING AFTER THE VERTICAL LINE IS YOUR ANSWER! SO:

$x = 1,\ y = -2,\ z = 0.$