Practice Problems

MATH2055: Advanced Linear Algebra Tutorial 5

Fun with Isomorphisms

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April 4, 2024

Question 1 - Find An Isomorphism

(Anton 8.3.9 and 8.3.10)

1. Show that $\mathbb{P}_2 \simeq \mathbb{R}^3$ by finding an isomorphism.
2. Let S_3 be the vector space of all 3×3 symmetric matrices. Show that $S_3 \simeq \mathbb{R}^6$ by finding an isomorphism.
3. Consider the vector space $V := \text{span}\{1, \sin t, \cos t\}$. Show that $V \simeq \mathbb{R}^3$ by finding an isomorphism.

Question 2 - Isomorphism is Transitive

(Anton 8.3.23) Prove that if U V, and W are vector spaces such that $U \simeq V$ and $V \simeq W$ then $U \simeq W$.

Question 3 - Inner Product Space Isomorphisms

(Anton 8.3.20) We know that $\mathbb{M}_{2\times 2}$ (the 2×2 matrices with real number entries) is an inner product space where $\langle A, B\rangle = \operatorname{tr}\left(B^\top A\right)$. Also, we know $\mathbb{P}_3$ is an inner product space where

$\langle a_0 + a_1x + a_2x^2 + a_3x^3, b_0 + b_1x + b_2x^2 + b_3x^3\rangle =$
$a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3$.

We say two inner product spaces are isomorphic if there exists some vector space isomorphism T such that $\langle T(u), T(v)\rangle = \langle u, v\rangle$. Show that $T : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to a + bx + cx^2 + dx^3$ is an inner product space isomorphism from $\mathbb{M}_{2\times 2}$ to $\mathbb{P}_3$.