THIS WEEK, WE'LL TALK ABOUT CHANGE OF BASIS. WE WILL DISCUSS 3 TOPICS:

I) WRITING A COORDINATE VECTOR FOR A GIVEN BASIS.

SAY $v = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ AND $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

THEN
$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ 0 \\ \alpha_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha_2 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} \alpha_3 \\ \alpha_3 \\ 0 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_3 \\ \alpha_2 + \alpha_3 \\ \alpha_1 + \alpha_2 \end{bmatrix} \longrightarrow \text{SYSTEM OF EQUATIONS!}$$
$$\Rightarrow \begin{cases} \alpha_1 + \alpha_3 = 3 \\ \alpha_2 + \alpha_3 = 2 \\ \alpha_1 + \alpha_2 = 1 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

SO $[v]_B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ (ie $v = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$)

II) FINDING $[I]_{B \leftarrow A}$.

SAY $A = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \right\}$

AND $B = \left\{ \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} \right\}$

BOTH BASES FOR $M_{2\times 2}$

WRITE EACH MATRIX IN B USING THE BASIS A. (USING I.)

$$\left[\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}\right]_A = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad \left[\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\right]_A = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$
$$\left[\begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix}\right]_A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \left[\begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}\right]_A = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

PUTTING IT TOGETHER:
$$[I]_{B \leftarrow A} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

III) FINDING $[T]_{B\leftarrow A}$

SAY $T: V \to W$, WHERE A IS A BASIS OF V AND B IS A BASIS OF W.
WE CAN USE A COMMUTATIVE DIAGRAM TO FIND $[T]_{B\leftarrow A}$ IN A FEW STEPS.

$$
\begin{tikzcd}
A & V \arrow[r, "{[T]_{B\leftarrow A}}"] \arrow[d, "{[I]_{S\leftarrow A}}"'] \arrow[dr, phantom, "\circlearrowleft"] & W & B \\
S & V \arrow[r, "{[T] = [T]_{S\leftarrow S}}"'] & W \arrow[u, "{[I]_{B\leftarrow S}}"'] & S
\end{tikzcd}
$$

WHAT DOES A COMMUTATIVE DIAGRAM MEAN?
IT DOES NOT MATTER WHAT PATH YOU TAKE!
SO $[T]_{B\leftarrow A} = [I]_{B\leftarrow S}[T]_{S\leftarrow S}[I]_{S\leftarrow A}$

WE FIND $[I]_{B\leftarrow S}$ AND $[I]_{S\leftarrow A}$ LIKE WE DID IN II.
WE FIND $[T]_{S\leftarrow S}$ LIKE WE DID A WHILE BACK.