THIS TIME, WE'LL TALK ABOUT AN "ENHANCED" TYPE OF VECTOR SPACE.

AN INNER PRODUCT SPACE V IS A VECTOR SPACE ALONG WITH A PRODUCT, CALLED AN INNER PRODUCT, $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$, SUCH THAT:

(a) $\langle u, v \rangle = \overline{\langle v, u \rangle}$ CONJUGATE SYMMETRY

(b) $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ ADDITIVITY IN 1ST SLOT

(c) $\langle ku, v \rangle = k\langle u, v \rangle$ HOMOGENEITY IN 1ST SLOT

$\}$ LINEARITY IN 1ST SLOT

(d) $\langle v, v \rangle \geq 0$ AND $\langle v, v \rangle = 0$ IFF $v = 0$ POSITIVE DEFINITENESS

FROM THE INNER PRODUCT, WE DEFINE THE NORM OF A VECTOR $v \in V$ AS:

$\|v\|^2 = \langle v, v \rangle$

THEN WE CAN DEFINE AN ABSTRACT VERSION OF ANGLE:

$\langle u, v \rangle = \|u\| \|v\| \cos \theta$

WE CAN ALSO DEFINE AN ABSTRACT VERSION OF DISTANCE, CALLED A METRIC

$d(u, v) = \|u - v\| = \sqrt{\langle u-v, u-v \rangle}$

QUICK NOTE: WE HAVE CONJUGATE SYMMETRY IN GENERAL. FOR REAL INNER PRODUCT SPACES, WE HAVE FULL SYMMETRY.

WHAT IS A CONJUGATE?

IF $z = a + ib$, $\bar{z} = a - ib$.

SO JUST SWAP SIGN OF IMAGINARY PART.

TWO KEY INEQUALITIES FOR NORMS:

(1) CAUCHY-SCHWARZ INEQUALITY

$|\langle u, v \rangle| \leq \|u\| \|v\|$

(2) TRIANGLE INEQUALITY

$\|u+v\| \leq \|u\| + \|v\|$

AS PRACTICE, WE WILL SHOW THE REVERSE TRIANGLE INEQUALITY:

$\big| \|v\| - \|u\| \big| \leq \|v - u\|$

Pf. LET u, v BE VECTORS IN AN INNER PRODUCT SPACE V. THEN WE KNOW Δ INEQUALITY HOLDS. SO $\|u\| + \|v - u\| \geq \|u + v - u\| = \|v\|$ AND $\|v\| + \|u - v\| \geq \|u\|$ WLOG. REARRANGING YIELDS $\|v - u\| \geq \|v\| - \|u\|$, AND $\|u - v\| \geq \|u\| - \|v\|$. NOTICE THAT $\|u - v\| = \|v - u\|$ $(= d(u, v))$ AND SO LET'S TAKE THE ABSOLUTE VALUE OF BOTH SIDES TO GET $\|v - u\| \geq \big| \|v\| - \|u\| \big|$. QED

SOME COMMON INNER PRODUCTS

ON $\mathbb{R}^n$: $\langle v, u \rangle = \sum_i v_i u_i$

ON $\mathbb{C}^n$: $\langle v, u \rangle = \sum_i v_i \overline{u_i}$

ON $\mathbb{R}^{\mathbb{R}}$: $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) g(x)\, dx$